Lecture 6

Source: Lee, First Course in Combinatorial optimization

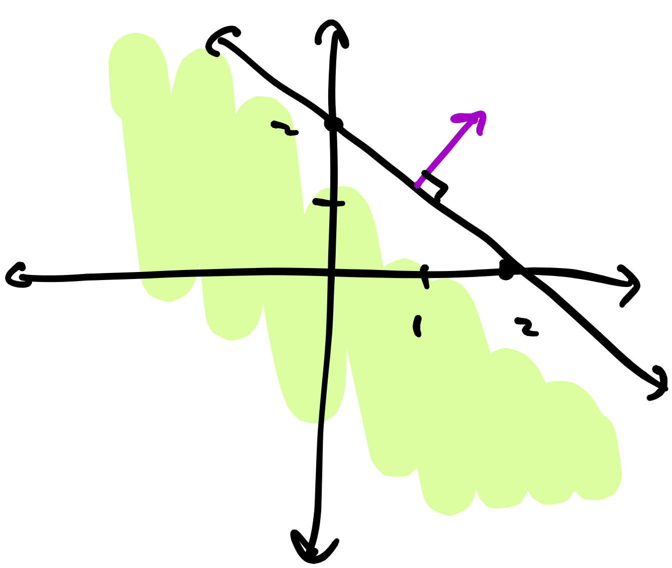
Plan:

1. Definitions
2. Solvability of systems of inequalities.
3. Linear programming duality.

Definitions

Def (Halfspace): set $\{x \in \mathbb{R}^n : a^T x \leq b\}$

$a \in \mathbb{R}^n$ $\quad$ $b \in \mathbb{R}$



$a = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$b = 2.$

Def (Polyhedron): Intersection of finitely many half spaces. write

$$P = \{x \in \mathbb{R}^n : Ax \leq b\}$$

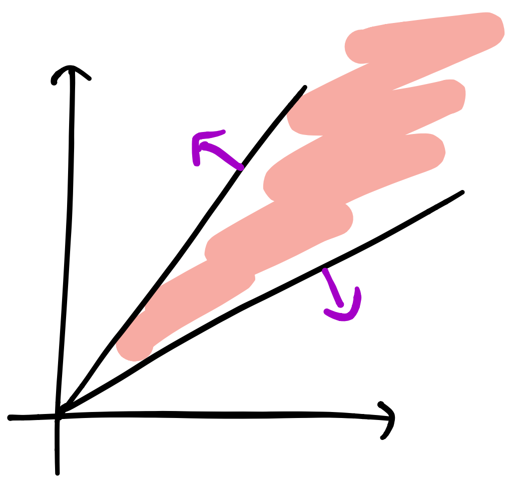
A: $m \times n$ matrix

$\leq$: coordinatewise

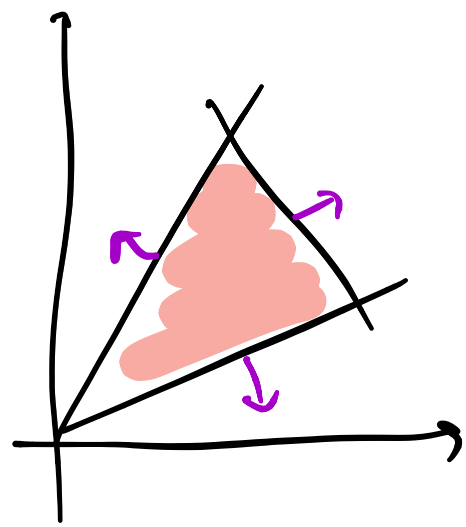
b: $\in \mathbb{R}^m$

$$\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \leq \begin{pmatrix} 2 \\ 1.5 \\ 3.5 \end{pmatrix}$$

Def (Polytope): Bounded polyhedron.



polyhedron

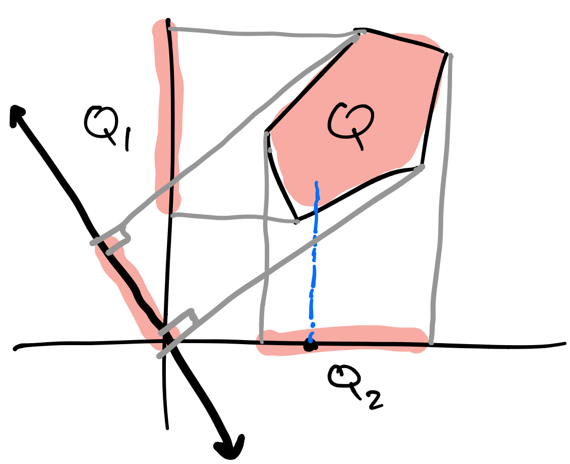


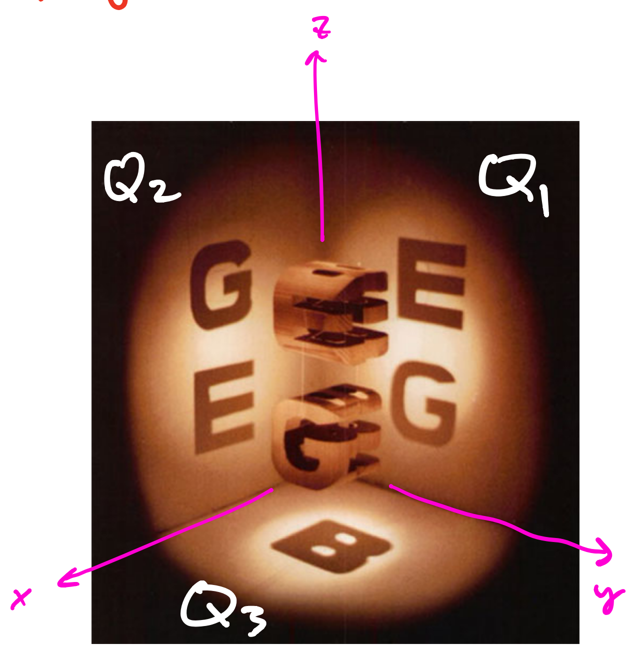
polytope.

Def: If $Q \subseteq \mathbb{R}^n$ is a set, (coordinate) <u>projection</u> $Q_k \subseteq \mathbb{R}^{n-1}$ is

$$Q_k := \{(x_1, \dots x_{k-1}, x_{k+1}, \dots x_n) : x \in Q \text{ for some } x_k \in \mathbb{R}\}.$$

- (is some way to choose x_k to "complete")
- special case of projection to subspace.





Claim: P polyhedron $\Rightarrow P_k$ polyhedron.

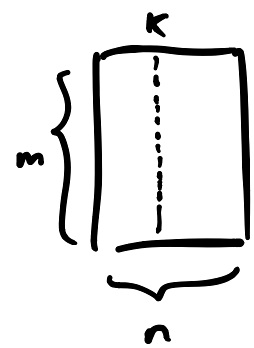
Proof: Give P_k's inequalities.

Method:

Fourier-Motzkin elimination

Let $P = \{x : Ax \leq b\}$ elim x_k

- $S_+ = \{i : a_{ik} > 0 \}$



- $S_- = \{ i : a_{ik} < 0 \}$

- $S_0 = \{ i : a_{ik} = 0 \}$.

$S_0, S_-, S_+ \subseteq \{1, \ldots\ldots, m\}$

E.g. $n=3, m=4, k=2$.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 3 & 1 \\ -4 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$S_+ = \{1,2\}$

$S_- = \{3\}$

$S_0 = \{4\}$.

- Any x in P_k satisfies

  (*) $\quad a_i^T x \le b_i$

  for all $i \in S_0$.

  (because these don't involve x_k ($a_{ik}=0$))

- Can take linear combination of S_+, S_- ineqs to eliminate coefficient of x_k:

if $i \in S_+$, $l \in S_-$,

(**) $\boxed{\begin{array}{l} a_\ell^T x \le b_\ell \qquad a_i^T x \le b_i \\ a_{ik}(a_\ell^T x) - a_{\ell k}(a_i^T x) \le a_{ik} b_\ell - a_{\ell k} b_i \end{array}}$

for all $x \in P_k$.

coeff of x_k: $a_{ik} a_{\ell k} - a_{\ell k} a_{ik} = 0$.

Claim: The inequalities

$$\left\{ \begin{array}{l} *: i \in S_0, \\ **: i \in S_+, \ell \in S_- \end{array} \right\}$$

describe P_k. $|S_0| + |S_+||S_-|$

b/c *, ** don't involve x_k, are satisfied for $x \in P_k$.

Still Need to show:

for any $(x_1, \dots, x_{k-1}, x_{k+1}, \dots x_n)$

satisfying (*) and (**)

there is x_k s.t. $x \in P$. Why?

(i) for $i \in S_+$, $a_i^T x \le b_i$

is upper bound on x_k.

$$a_{ik} x_k + \sum_{j \neq k} a_{ij} x_j \le b_i$$

$$\Longleftrightarrow \quad x_k \le \boxed{\frac{b_i - \sum_{j \neq k} a_{ij} x_j}{a_{ik}}} \quad (U)$$

(ii) for $l \in S_-$, is lower bd.

$$a_{lk} x_k + \sum_{j \neq k} a_{lj} x_j \le b_l$$

$$\Longleftrightarrow \quad x_k \ge \boxed{\frac{b_l - \sum_{j \neq k} a_{lj} x_j}{a_{lk}}} \quad (L)$$

(iii) $(**)$ says

every such upper bound (U) on x_k

is bigger than lower bound (L) on x_k.

$$(u) \leq (L) \iff$$

$$\frac{b_\ell - \sum_{j \neq k} a_{\ell j} x_j}{a_{\ell k}} \leq \frac{b_i - \sum_{j \neq k} a_{ij} x_j}{a_{ik}}$$

$$\iff$$

$$a_{jk} \left(\frac{b_\ell - \sum_{j \neq k} a_{\ell j} x_j}{a_{\ell k}} \right) \leq b_i - \sum_{j \neq k} a_{ij} x_j$$

$$\iff$$

$$a_{ik} \left(b_\ell - \sum_{j \neq k} a_{\ell j} x_j \right) \geq a_{\ell k} \left(b_i - \sum_{j \neq k} a_{ij} x_j \right)$$

$$\iff \ast \ast .$$

$\Rightarrow \exists$ some $x_k \in \mathbb{R}$ satisfying all inequalities in S_-, S_+. $\square$

Summary of FM elim:

$\boxed{\begin{array}{l} \text{system } Ax \leq b \\ \rightsquigarrow \text{new system } \tilde{A}x \leq \tilde{b} \end{array}}$

Properties:

(1) New inequalities don't involve x_k

$$A = \begin{array}{c c} & n \\ m & \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \end{array} \qquad \tilde{A} = \begin{array}{c c} & n \\ \tilde{m} & \begin{bmatrix} & 0 & \\ & 0 & \\ & 0 & \end{bmatrix} \\ & k \end{array}$$

(2) Inequalities of $\tilde{A}x \leq \tilde{b}$

are nonnegative linear combinations of those in $Ax \le b$.

$$\tilde{a}_i^T x \le \tilde{b}_i = \sum_j y_j (a_j^T x \le b_j)$$

for some $y_j \ge 0$.

(3) $Ax \le b \Rightarrow \tilde{A}x \le \tilde{b}$.

(4) $\tilde{A}x \le \tilde{b} \Rightarrow \exists y$ s.t.

$$A(x_1, \ldots, x_{k-1}, y, x_{k+1}, \ldots x_n) \le b.$$

(3),(4) $\Rightarrow$ $\boxed{\begin{array}{c} Ax \le b \text{ solvable} \\ \Longleftrightarrow \\ \tilde{A}x \le \tilde{b} \text{ solvable.} \end{array}}$

MORE DEFS

Def: for $a^{(1)}, \ldots, a^{(k)} \in \mathbb{R}^n$,

<u>linear</u> combination:

$$\sum_{i=1}^{k} \lambda_i a^{(i)} \qquad \lambda_i \in \mathbb{R}$$

<u>affine</u> combination:

$$\sum_{i=1}^{k} \lambda_i a^{(i)} \qquad \sum_{i=1}^{k} \lambda_i = 1$$

<u>conical</u> combination:

$$\sum_{i=1}^{k} \lambda_i a^{(i)} \qquad \lambda_i \geq 0$$

<u>convex</u> combination:

$$\sum_{i=1}^{k} \lambda_i a^{(i)} \quad \lambda_i \geq 0, \ \sum \lambda_i = 1$$

(affine + conical)

<u>linear hull</u>: lin(S) = all linear combinations of elements of S.

↳ span(S).

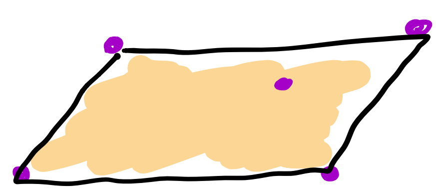
<u>affine hull</u>: aff(S) = " affine

<u>conical</u> " cone(S) = " conical

• <u>Convex</u> " $\boxed{\text{conv}(S)}$ = " convex.

$a^T x = b$

$\Leftrightarrow a^T x \leq b$ & $-a^T x \leq -b$



Def (Equiv. def. of polytope):

A polytope is the convex hull of finitely many points.

Why equivalent? $S = \{a^{(1)}, \dots, a^{(k)}\} \in \mathbb{R}^n$

$P = \mathrm{conv}(S) \Rightarrow P$ bounded polyhedron:

① P polyhedron: P is projection of polyhedron $\tilde{P}$ in $\mathbb{R}^{n+k}$:

$$\tilde{P} = \left\{ (\underbrace{x}_{n}, \underbrace{\lambda_1 \dots \lambda_k}_{k}) : \begin{array}{l} x - \sum_k \lambda_k a^{(k)} = 0 \\ \sum_k \lambda_k = 1 \\ \lambda_k \geq 0. \\ x \in \mathbb{R}^n \end{array} \right\}$$

project out last k coords. of $\tilde{P}$ to get P.

we already saw: proj. of polyhedron is polyhedron! (Fourier-Motzkin)

② **P bounded**: convex combos have

$$\left\|\sum \lambda_k a^{(k)}\right\| \le \sum \left\|\lambda_k a^{(k)}\right\|$$

$$\le \sum \lambda_k \left\|a^{(k)}\right\| \le \max_k \left\|a^{(k)}\right\|.$$

($\Leftarrow$)? Later in notes.

Solvability of Systems of Inequalities

Linear algebra: $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$;

$$\boxed{\begin{array}{l} Ax = b \text{ has no solution} \\ \iff \exists\, y \in \mathbb{R}^m : A^T y = 0,\ b^T y \neq 0 \end{array}}$$

Why? $\mathrm{Col}(A) = \{\text{possible } b\text{'s}\} = \mathrm{Null}(A^T)^\perp$.

A form of duality: y's obstruct x's.

$\Leftarrow\ 0 \neq b^T y = (Ax)^T y = x^T A^T y = 0.$

contradiction.

For inequalities:

Theorem (Theorem of the Alternatives) (TOTA).

$$\boxed{Ax \le b \text{ has no solution} \iff \exists\, y \in \mathbb{R}^m : A^T y = 0,\ b^T y < 0,\ y \ge 0}$$

Proof: ($\Leftarrow$) simplest: Suppose $Ax \le b$; then

$$\boxed{0 > b^T y \underset{\substack{\uparrow \\ Ax \le b,\ y \ge 0.}}{\ge} (Ax)^T y = x^T A^T y = 0.}$$

Contradiction.

($\Rightarrow$) Fourier-Motzkin Elim.

- Eliminate **all** variables - get

  $\tilde{A}x \le \tilde{b}$ unsolvable

- $\tilde{A} = 0$ (m×n zero matrix)

- $0x \le \tilde{b}$ is unsolvable.

  $\iff$ ~~some~~ $\exists i$ s.t. $\tilde{b}_i < 0$.

- But then $0^T x \le b_i$

  is a nonnegative linear combo:

  $$\underbrace{\sum y_i (\underbrace{a_i^T x}_{} \le b_i) = (\underbrace{0^T x}_{} \le b_i)}_{}$$

  i.e. $\boxed{A^T y = 0,\ y \ge 0,\ b^T y < 0}$ $\square$

  $b^T y = b_i < 0$

Variant (mixed $=/\leq$)

Example:

$$\underbrace{\begin{aligned} a_1^T x &\leq b_1 \\ a_2^T x &\leq b_2 \\ a_3^T x &= b_3 \end{aligned}}_{Ax \,\triangle\, b} \text{ has no soln} \iff \exists\, y \text{ s.t. } A^T y = 0,\ b^T y < 0, \text{ and } \underbrace{\begin{aligned} y_1 &\geq 0 \\ y_2 &\geq 0 \\ y_3 &\in \mathbb{R} \end{aligned}}_{y \,\square\, 0}$$

$\triangle \rightsquigarrow \square$: replace $= \rightsquigarrow$ unconstrained.

$\leq \rightsquigarrow \geq$.

Ex: prove variant.

Another variant:

Farkas Lemma:

$Ax = b$,

has no solution

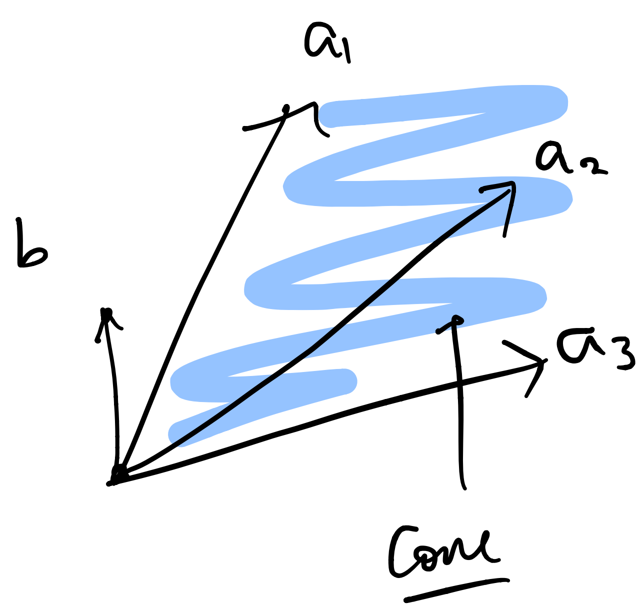
$x \geq 0$ has no ...

$\Leftrightarrow \exists y$ s.t. $A^T y \geq 0$, $b^T y < 0$.

Ex. prove Farkas from TOTA.

Picture: $A = \begin{bmatrix} | & & | \\ a_1 & \cdots & a_n \\ | & & | \end{bmatrix}$

e.g.



Ax, $x \geq 0$.

separating hyperplane theorem.

LP Duality

Linear "program": maximizing linear function over polyhedron.

max : $\underbrace{c^T x}_{\text{objective}}$

(P)

subject to : $\underbrace{Ax \le b}_{\text{constraints.}}$

- x **feasible** if satisfies constraints.
- If no x is feasible, say (P) **infeasible**,
- If value $+\infty$, say (P) **unbounded**. Else (P) **bounded**.
- value finite $\Leftrightarrow$ (P) neither infeas. nor unbounded.
- Many equivalent forms also LPs, e.g. with constraints $\geq, =, \leq$.

e.g. min weight perfect matching

$$\min \{ c^T x : Ax = b, x \geq 0 \}.$$

The Dual of (P):

$$\text{(D)} \quad \begin{aligned} \min:\ & b^T y \\ \text{subject to:}\ & A^T y = c \\ & y \geq 0. \end{aligned}$$

- (D) said to be **dual**, (P) **primal**.
- Terminology for (D) analogous except (D) **unbounded** if value is $-\infty$
  (D) **infeas** if value $+\infty$

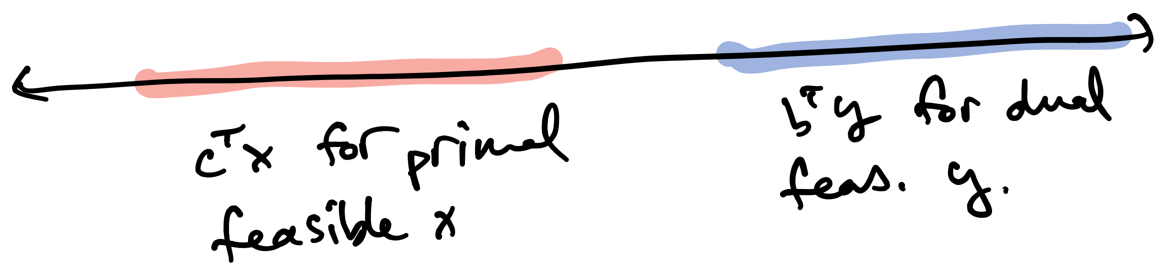
Note: primal/dual vars different: if $A \in \mathbb{R}^{m \times n}$, then

n primal vars, m dual vars

Weak duality: For feasible solns x, y to (P), (D),

$$c^T x \le b^T y.$$

Picture:



Proof: $c^T x = y^T A x \le y^T b = b^T y$

$A^T y = c$ $\qquad$ $Ax \le b$, $y \ge 0$

The dual was defined this way precisely so this would happen.

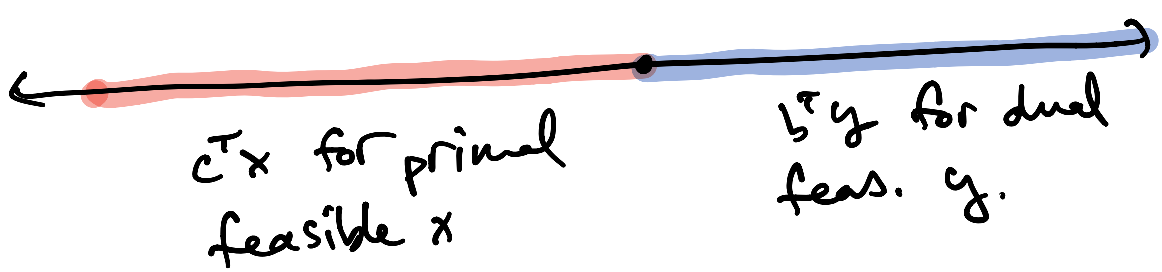
Corollary:

(P) unbounded $\Rightarrow$ (D) infeasible

(D) unbounded $\Rightarrow$ (P) infeasible.

Theorem (Strong Duality)

Suppose (P), (D) feasible.
Then optimal values are the same.



Many proofs: e.g.

- von Neumann minimax theorem
- Fourier-Motzkin elim. ↙ latex notes.

TODAY: Proof using TOTA.

IDEA: write down bigger system encoding

(i) x primal feasible

(ii) y dual feasible

(iii) $c^T x \geq b^T y$.

Use TOTA to show feasible.

Proof: Suppose x^* feasible for (P), y^* feasible for (D).

For contradiction, assume following is <u>infeasible</u>:

<u>System</u>:

$$\begin{aligned}
Ax \quad\quad\quad\quad &\leq b && \text{(i)} \\
A^T y &= c && \\
-Iy &\leq 0 && \text{(ii)} \\
-c^T x + b^T y &\leq 0 && \text{(iii)}
\end{aligned}$$

gives some polyhedron $\{\tilde{x} : \tilde{A}\tilde{x} \leq \tilde{b}\}$ for

$\tilde{x} \in \mathbb{R}^{m+n}$

Matrix:

$$\underbrace{\begin{bmatrix} A & 0 \\ 0 & A^T \\ 0 & -I \\ -c^T & b^T \end{bmatrix}}_{\tilde{A}} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\tilde{x}} \underset{\Delta}{\begin{matrix} \le \\ = \\ \le \\ \le \end{matrix}} \underbrace{\begin{bmatrix} b \\ c \\ 0 \\ 0 \end{bmatrix}}_{\tilde{b}}$$

TOTA: $\tilde{A}\tilde{x} \,\Delta\, \tilde{b}$ infeasible $\iff$

$\exists\ \tilde{y}$ s.t. $\boxed{\underbrace{\tilde{A}^T\tilde{y} = 0}_{(1)},\ \underbrace{\tilde{y} \,\square\, 0}_{(2)},\ \underbrace{\tilde{b}^T\tilde{y} < 0}_{(3)}.}$

$$\tilde{A}^T \qquad \tilde{y} \qquad (1)$$

$$\left[\begin{array}{c|c|c|c} A^T & 0 & 0 & -c \\ \hline 0 & A & -I & b \end{array}\right] \begin{bmatrix} s \\ \hline t \\ \hline u \\ \hline v \end{bmatrix} = 0$$

$v \rightarrow$ a number.

,

$$\tilde{y} = \begin{bmatrix} s \\ \hline t \\ \hline u \\ \hline v \end{bmatrix} \begin{array}{l} \geq 0 \\ ? \ 0 \\ \geq 0 \\ \geq 0 \end{array} \qquad (2)$$

$$\underbrace{\left[\, b^T \mid c^T \mid 0 \mid 0 \,\right]}_{\tilde{b}^T} \underbrace{\begin{bmatrix} s \\ \hline t \\ \hline u \\ \hline v \end{bmatrix}}_{\tilde{y}} < 0. \qquad (3)$$

Writing these out:

$$
* \begin{cases}
A^T s - vc = 0 \\
At - u + vb = 0 \\
b^T s + c^T t < 0. \\
s \geq 0 \\
u \geq 0 \\
v \geq 0.
\end{cases}
$$

$A^T s - vc = 0$ (1)

$b^T s + c^T t < 0$ (3)

$s \geq 0$, $u \geq 0$ (2)

Case 1: $v = 0.$

$\Rightarrow A^T s = 0$

$\Rightarrow y^* + \alpha s$ dual feas.

for all $\alpha \geq 0$.

$\left(A^T(y^* + \alpha s) = A^T y^* = c\right)$

Similarly,

$At \geq 0$ $\quad (At = u \geq 0)$

$\Rightarrow x^* - \alpha t$ primal feasible, $\alpha \geq 0$

By weak duality,

$$c^T(x^* - \alpha t) \leq b^T(y^* + \alpha s).$$

$$\Longleftrightarrow$$

$$c^T x^* - b^T y^* \leq \alpha \underbrace{(b^T s + c^T t)}_{<0}$$

$\alpha \to +\infty$

(3)

contradiction!

RHS $\to -\infty$, LHS fixed.

Case 2 $\quad v > 0.$

Recall: $A^T s - vc = 0$

$At - u + vb = 0$

$b^T s + c^T t < 0.$

$s \geq 0$

$u \geq 0$

$v \geq 0.$

Divide through by v,

Rename $\frac{s}{v} \leftarrow s$, $\frac{t}{v} \leftarrow t$, $\frac{u}{v} \leftarrow u$

get $A^T s - c = 0$

$At - u = -b$

$b^T s + c^T t < 0$

$s \geq 0$

$u \geq 0$

$\Rightarrow$ s dual feasible,

$-t$ primal feasible. $A(-t) = b - u \leq b$

$\Rightarrow$ $c^T(-t) \leq b^T s$.

$b^T s + c^T t \geq 0$ $\square$

contradicts $\supset$.

Ex. Dual of dual is primal.

Ex. Strong duality holds when either (P) or (D) feas.

i.e. if (P) feas but dual infeas, then (P) unbounded. (both values $+\infty$).

Ex: find example where both infeasible.

$-\infty$ primal $\longleftrightarrow$ $+\infty$ dual

What do optimal solutions

$$x^*, \; y^*$$

look like? look at

$$\boxed{c^T x = y^T A x \le b^T y}$$

equal only if when $(Ax)_i < b_i$, $y_i = 0$.

Theorem (Complementary Slackness)

Suppose x primal feas., y dual feas.

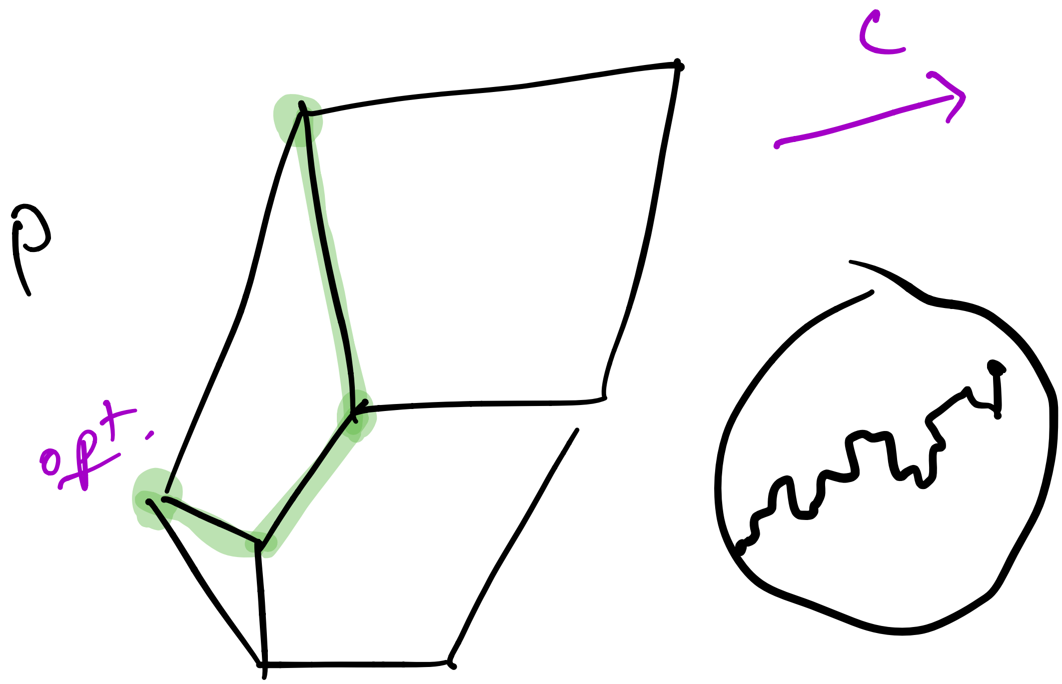
Then

$$\boxed{\begin{array}{l} x \text{ optimum in } (P) \\ y \text{ optimum in } (D) \end{array}}$$

$$\Longleftrightarrow$$

$$\boxed{\begin{array}{l} \forall i \text{ either } y_i = 0 \\ \quad \text{or } (Ax)_i = b_i \\ \quad \text{or both.} \end{array}}$$

Simplex method:



Doesn't provably run in polynomial time.

There *are* poly-time algorithms:

- ellipsoid
- interior point methods.